REFERENCE

1.2 OPERATIONS ON FUNCTIONS AND TYPES OF FUNCTIONS

New functions may be formed from given functions by adding, subtracting, multiplying, and dividing function values. Accordingly, these new functions are known as the *sum, difference, product,* and *quotient* of the original functions.

1.2.1 Definitions of the Sum, Difference, Product, and Quotient of Two Functions

Given the two functions f and g:

(i) their **sum**, denoted by $f + g$, is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

(ii) their **difference**, denoted by $f - g$, is the function defined by

$$(f - g)(x) = f(x) - g(x)$$

(iii) their **product,** denoted by $f \cdot g$, is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

(iv) their **quotient,** denoted by f/g, is the function defined by

$$(f/g)(x) = f(x)/g(x) \qquad g(x) \neq 0$$

In each case the *domain* of the resulting function consists of those values of x common to the domains of f and g, with the additional requirement in case (iv) that the values of x for which $g(x) = 0$ are excluded.

EXAMPLE 1 Given that f and g are the functions defined by

$$f(x) = \sqrt{x+1} \quad \text{and} \quad g(x) = \sqrt{x-4}$$

define the following functions and determine the domain of the resulting function: **(a)** $f + g$; **(b)** $f - g$; **(c)** $f \cdot g$; **(d)** f/g.

Solution

(a) $(f + g)(x) = \sqrt{x+1} + \sqrt{x-4}$

(b) $(f - g)(x) = \sqrt{x+1} - \sqrt{x-4}$

(c) $(f \cdot g) = \sqrt{x+1} \cdot \sqrt{x-4}$

(d) $(f/g)(x) = \dfrac{\sqrt{x+1}}{\sqrt{x-4}}$

The domain of f is $[-1, +\infty)$, and the domain of g is $[4, +\infty)$. So in parts (a), (b), and (c) the domain of the resulting function is $[4, +\infty)$. In part (d) the denominator is zero when $x = 4$; thus 4 is excluded from the domain, and the domain is therefore $(4, +\infty)$. ◀

Obtaining the *composite function* of two given functions is another operation on functions.

1.2.2 Definition of a Composite Function

Given the two functions f and g, the **composite function,** denoted by $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x))$$

and the domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

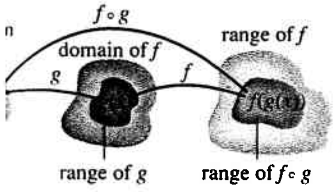


FIGURE 1

The definition indicates that when computing $(f \circ g)(x)$, we first apply function g to x and then function f to $g(x)$. To visualize this computation see Figure 1. Function g assigns the value $g(x)$ to the number x in the domain of g. Then function f assigns the value $f(g(x))$ to the number $g(x)$ in the domain of f. Observe in Figure 1 that the range of g is a subset of the domain of f and the range of $f \circ g$ is a subset of the range of f.

▷ **ILLUSTRATION 1** If f and g are defined by

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = 2x - 3$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x - 3) \\ &= \sqrt{2x - 3}\end{aligned}$$

The domain of g is $(-\infty, +\infty)$, and the domain of f is $[0, +\infty)$. The domain of $f \circ g$ is, therefore, the set of real numbers x for which $2x - 3 \geq 0$ or, equivalently, $[\frac{3}{2}, +\infty)$. ◀

▶ ***EXAMPLE 2*** Given

$$f(x) = \frac{5}{x - 2} \quad \text{and} \quad g(x) = 2x + 1$$

Compute $(f \circ g)(3)$ by two methods: **(a)** Find $g(3)$ and use that number to find $f(g(3))$; **(b)** Compute $(f \circ g)(x)$ and use that value to find $(f \circ g)(3)$.

Solution

(a)

$$\begin{aligned}g(3) &= 2(3) + 1 \\ &= 7\end{aligned}$$

Thus

$$\begin{aligned}f(g(3)) &= f(7) \\ &= \frac{5}{7 - 2} \\ &= 1\end{aligned}$$

(b)

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x + 1) \\ &= \frac{5}{(2x + 1) - 2} \\ &= \frac{5}{2x - 1}\end{aligned}$$

Therefore

$$\begin{aligned}(f \circ g)(3) &= \frac{5}{2(3) - 1} \\ &= 1\end{aligned}$$

◀

▶ ***EXAMPLE 3*** Given that f and g are defined by

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = x^2 - 1$$

find: **(a)** $f \circ f$; **(b)** $g \circ g$; **(c)** $f \circ g$; **(d)** $g \circ f$. Also determine the domain of the composite function in each part.

Solution The domain of f is $[0, +\infty)$ and the domain of g is $(-\infty, +\infty)$.

(a)

$$\begin{aligned}(f \circ f)(x) &= f(f(x)) \\ &= f(\sqrt{x}) \\ &= \sqrt{\sqrt{x}} \\ &= \sqrt[4]{x}\end{aligned}$$

The domain is $[0, +\infty)$.

(b)

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g(x^2 - 1) \\ &= (x^2 - 1)^2 - 1 \\ &= x^4 - 2x^2\end{aligned}$$

The domain is $(-\infty, +\infty)$.

(c) $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(x^2 - 1)\\ &= \sqrt{x^2 - 1}\end{aligned}$$

The domain is

$(-\infty, -1] \cup [1, +\infty)$.

(d) $$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(\sqrt{x})\\ &= (\sqrt{x})^2 - 1\\ &= x - 1\end{aligned}$$

The domain is $[0, +\infty)$.

In part (d) note that even though $x - 1$ is defined for all values of x, the domain of $g \circ f$, by the definition of a composite function, is the set of all numbers x in the domain of f such that $f(x)$ is in the domain of g. Thus the domain of $g \circ f$ must be a subset of the domain of f. ◀

Observe from the results of parts (c) and (d) of Example 3 that $(f \circ g)(x)$ and $(g \circ f)(x)$ are not necessarily equal.

An important theorem in calculus, called the ***chain rule,*** discussed in Section 2.8, involves composite functions. When applying the chain rule it is necessary to think of a function as the composition of two other functions, as shown in the following illustration.

▷ **ILLUSTRATION 2** If $h(x) = (4x^2 + 1)^3$, we can express h as the composition of the two functions f and g for which

$$f(x) = x^3 \qquad \text{and} \qquad g(x) = 4x^2 + 1$$

because

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(4x^2 + 1)\\ &= (4x^2 + 1)^3\end{aligned}$$ ◀

The function h in Illustration 2 can be expressed as the composition of other pairs of functions. For example, if

$$F(x) = (4x + 1)^3 \qquad \text{and} \qquad G(x) = x^2$$

then

$$\begin{aligned}(F \circ G)(x) &= F(G(x))\\ &= F(x^2)\\ &= (4x^2 + 1)^3\end{aligned}$$

▶ ***EXAMPLE 4*** Given

$$h(x) = \frac{1}{\sqrt{x^2 + 3}}$$

express h as the composition of two functions f and g in two ways: **(a)** the function f contains the radical; **(b)** the function g contains the radical.

Solution

(a) $f(x) = \dfrac{1}{\sqrt{x + 3}}$

$g(x) = x^2$

Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \\ &= \frac{1}{\sqrt{x^2 + 3}}\end{aligned}$$

(b) $f(x) = \dfrac{1}{x}$

$g(x) = \sqrt{x^2 + 3}$

Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x^2 + 3}) \\ &= \frac{1}{\sqrt{x^2 + 3}}\end{aligned}$$

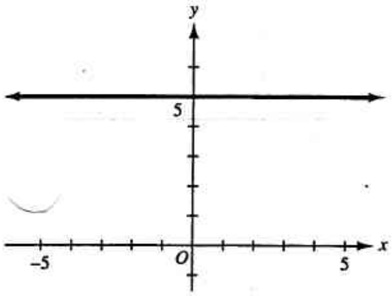
◀

A function whose range consists of only one number is called a **constant function.** Thus if $f(x) = c$, and if c is any real number, then f is a constant function, and its graph is a horizontal line at a directed distance of c units from the x axis.

▷ ILLUSTRATION 3

(a) The function defined by $f(x) = 5$ is a constant function, and its graph, sketched in Figure 2, is a horizontal line 5 units above the x axis.

(b) The function defined by $g(x) = -4$ is a constant function whose graph is a horizontal line 4 units below the x axis. See Figure 3. ◀



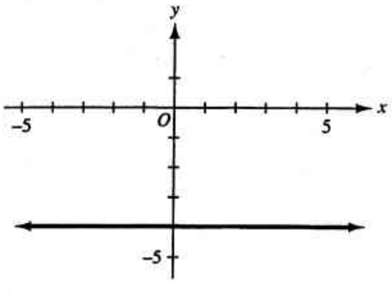
$f(x) = 5$

FIGURE 2

A **linear function** is defined by

$$f(x) = mx + b$$

where m and b are constants and $m \neq 0$. Its graph is a line having slope m and y intercept b.



$g(x) = -4$

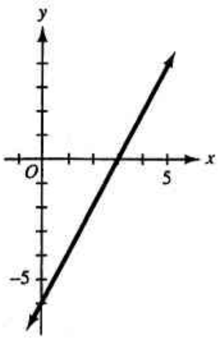
FIGURE 3

▷ ILLUSTRATION 4

The function defined by

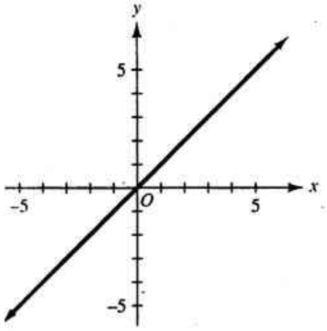
$$f(x) = 2x - 6$$

is linear. Its graph is the line shown in Figure 4. ◀



$f(x) = 2x - 6$

FIGURE 4



$f(x) = x$

FIGURE 5

The particular linear function defined by

$$f(x) = x$$

is called the **identity function.** Its graph, sketched in Figure 5, is the line bisecting the first and third quadrants.

If a function f is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers ($a_n \neq 0$) and n is a nonnegative integer, then f is called a **polynomial function** of degree n. Thus the function defined by

$$f(x) = 3x^5 - x^2 + 7x - 1$$

is a polynomial function of degree 5.

A linear function is a polynomial function of degree 1. If the degree of a polynomial function is 2, it is called a **quadratic function,** and if the degree is 3, it is called a **cubic function.**

If a function can be expressed as the quotient of two polynomial functions, it is called a **rational function.**

An **algebraic function** is one formed by a finite number of algebraic operations on the identity function and a constant function. These algebraic operations include addition, subtraction, multiplication, division, raising to powers, and extracting roots. Polynomial and rational functions are particular kinds of algebraic functions. A complicated example of an algebraic function is the one defined by

$$f(x) = \frac{(x^2 - 3x + 1)^3}{\sqrt{x^4 + 1}}$$

In addition to algebraic functions, we shall consider transcendental functions, examples of which are the trigonometric functions discussed in Appendix Section A.9 and logarithmic and exponential functions introduced in Chapter 5.

An *even function* is one whose graph is symmetric with respect to the y axis, and an *odd function* is one whose graph is symmetric with respect to the origin. Following are the formal definitions.

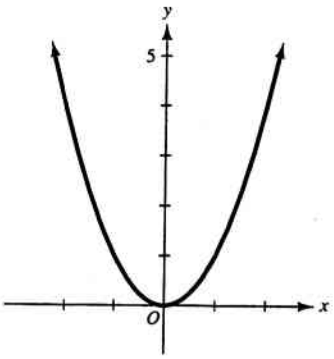
1.2.3 Definition of an Even and an Odd Function

(i) A function f is an **even function** if for every x in the domain of f, $f(-x) = f(x)$.

(ii) A function f is an **odd function** if for every x in the domain of f, $f(-x) = -f(x)$.

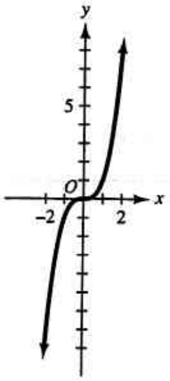
In both parts (i) and (ii) it is understood that $-x$ is in the domain of f whenever x is.

The symmetry properties of even and odd functions follow from the symmetry tests given in Appendix Section A.2.



$f(x) = x^2$

FIGURE 6



$g(x) = x^3$

FIGURE 7

ILLUSTRATION 5

(a) If $f(x) = x^2, f(-x) = (-x)^2$. Therefore $f(-x) = f(x)$, and f is an even function. Its graph is a parabola symmetric with respect to the y axis. See Figure 6.

(b) If $g(x) = x^3$, $g(-x) = (-x)^3$. Because $g(-x) = -g(x)$, g is an odd function. The graph of g, shown in Figure 7, is symmetric with respect to the origin.

EXAMPLE 5 Plot the graph of the given function and from the graph conjecture whether the function is even, odd, or neither; then confirm the conjecture analytically:

(a) $f(x) = 3x^4 - 2x^2 + 7$

(b) $g(x) = 3x^5 - 4x^3 - 9x$

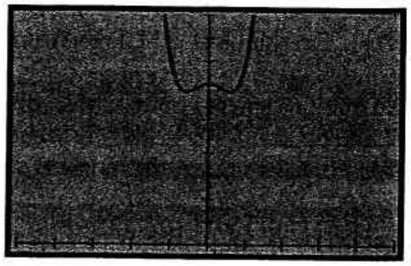
(c) $h(x) = 2x^4 + 7x^3 - x^2 + 9$

Solution

(a) The graph of f, plotted in Figure 8, appears symmetric with respect to the y axis. We, therefore, suspect the function is even. To prove this fact analytically, we compute $f(-x)$:

$$\begin{aligned} f(-x) &= 3(-x)^4 - 2(-x)^2 + 7 \\ &= 3x^4 - 2x^2 + 7 \\ &= f(x) \end{aligned}$$

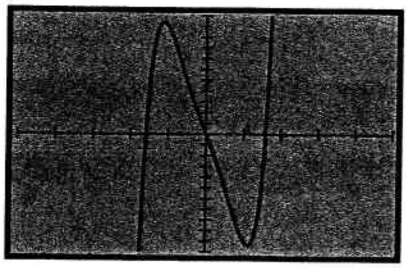
Because $f(-x) = f(x)$, f is even.



[-5, 5] by [0, 10]

$f(x) = 3x^4 - 2x^2 + 7$

FIGURE 8



[-5, 5] by [-11, 11]

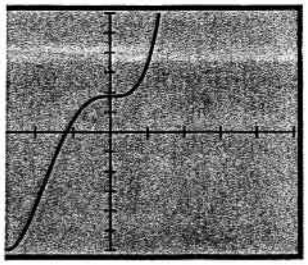
$g(x) = 3x^5 - 4x^3 - 9x$

FIGURE 9

(b) Figure 9 shows the graph of g apparently symmetric with respect to the origin. We suspect, therefore, that the function is odd. We compute $g(-x)$:

$$\begin{aligned} g(-x) &= 3(-x)^5 - 4(-x)^3 - 9(-x) \\ &= -3x^5 + 4x^3 + 9x \\ &= -(3x^5 - 4x^3 - 9x) \\ &= -g(x) \end{aligned}$$

Because $g(-x) = -g(x)$, we have proved analytically that g is odd.



[-5, 5] by [-30, 30]

$h(x) = 2x^4 + 7x^3 - x^2 + 9$

FIGURE 10

(c) Because the graph of h, appearing in Figure 10, is not symmetric w respect to either the y axis or the origin, the function is neither even odd. We compute $h(-x)$:

$$\begin{aligned} h(-x) &= 2(-x)^4 + 7(-x)^3 - (-x)^2 + 9 \\ &= 2x^4 - 7x^3 - x^2 + 9 \end{aligned}$$

Because $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we have confirmed tha is neither even nor odd.

▶ *EXAMPLE 6* Given

$$F(x) = |x + 3| - |x - 3|$$

(a) Define $F(x)$, without absolute value bars, piecewise in the follow intervals: $(-\infty, -3)$; $[-3, 3)$; $[3, +\infty)$. **(b)** Support the answer in part graphically by plotting the graph of F from the given equation. **(c)** From graph in part (b) state whether F is even, odd, or neither. **(d)** Confirm answer in part (c) analytically from the given equation.

Solution

(a) From the definition of the absolute value of a number

$$|x + 3| = \begin{cases} x + 3 & \text{if } x + 3 \geq 0 \\ -(x + 3) & \text{if } x + 3 < 0 \end{cases}$$

and

$$|x - 3| = \begin{cases} x - 3 & \text{if } x - 3 \geq 0 \\ -(x - 3) & \text{if } x - 3 < 0 \end{cases}$$

That is,

$$|x + 3| = \begin{cases} x + 3 & \text{if } x \geq -3 \\ -x - 3 & \text{if } x < -3 \end{cases}$$

and

$$|x - 3| = \begin{cases} x - 3 & \text{if } x \geq 3 \\ -x + 3 & \text{if } x < 3 \end{cases}$$

If $x \in (-\infty, -3)$, $|x + 3| = -x - 3$ and $|x - 3| = -x + 3$. Hence

$$\begin{aligned} |x + 3| - |x - 3| &= -x - 3 - (-x + 3) \\ &= -6 \end{aligned}$$

If $x \in [-3, 3)$, $|x + 3| = x + 3$ and $|x - 3| = -x + 3$. Thus

$$\begin{aligned} |x + 3| - |x - 3| &= x + 3 - (-x + 3) \\ &= 2x \end{aligned}$$

If $x \in [3, +\infty)$, $|x + 3| = x + 3$ and $|x - 3| = x - 3$. Theref

$$\begin{aligned} |x + 3| - |x - 3| &= x + 3 - (x - 3) \\ &= 6 \end{aligned}$$

With these results, we define $F(x)$ piecewise as follows:

$$F(x) = \begin{cases} -6 & \text{if } x < -3 \\ 2x & \text{if } -3 \leq x < 3 \\ 6 & \text{if } 3 \leq x \end{cases}$$